CONVEX HULL OF BROWNIAN MOTION IN d-DIMENSIONS*

BY

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ABSTRACT

We suppose $K(w)$ to be the boundary of the closed convex hull of a sample path of $Z_t(w)$, $0 \leqq t \leqq 1$ of Brownian motion in d-dimensions. A combinatorial result of Baxter and Borndorff Neilson on the convex hull of a random walk, and a limiting process utilizing results of P. Levy on the continuity properties of $Z_t(w)$ are used to show that the curvature of $K(w)$ is concentrated on a metrically small set.

Denote by $Z(t, \omega)$ the Brownian motion, starting at the origin, in real Euclidean d dimensional space. Let $J(\omega)$ be the convex hull of $Z(t,\omega)$, $0 \leqq t \leqq 1$, and let $K(\omega)$ be the boundary of $J(\omega)$.

Definition. Let $h(t)$ be a monotone positive continuous function with $h(0) = 0$. Let

$$h_\rho(E) = \text{g.l.b.} \sum_i h(\text{diam}\, 0_i)$$

where $\{0_i\}$ is a set of spheres covering E with diameter of 0_i less than ρ, and the greatest lower bound is taken over such coverings. The h-measure of E is defined by

$$h^*(E) = \lim_{\rho \to 0} h_\rho(E).$$

For a discussion of such measures see [2].

Theorem. *Let $h(t)$ be defined as above and satisfy*

$$\lim_{t \to 0} h(t)[\log(1/t)]^{d-1} = 0. \tag{1}$$

Then, for almost all ω, the total curvature of $K(\omega)$ is concentrated on a set $T(\omega)$ with

$$h^*(T(\omega)) = 0.$$

This generalizes the result of [3].

Proof. The proof rests on the following result of Baxter and Barndorff-

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Nielson [1]. Let X_i, $i = 1,2,\cdots$ be independent, identically distributed d dimensional vectors with uniform angular distribution. Let $S_0 = 0$, $S_i = \Sigma_{k \leqq i} X_k$. If H_m is the number of $d-1$ faces in the boundary of the convex hull of $\{S_0, \cdots, S_m\}$, then the expectation of H_m,

$$E\{H_m\} = A_1(\log m)^{d-1}$$

where A_1 depends on d but not on m.

Since each face of K_m has at least d vertices, the number F_m of vertices must satisfy

$$E(F_m) \leqq A_2(\log m)^{n-1} \tag{2}$$

where A_2 is independent of m.

Let $J_n(\omega)$ be the convex hull of $\{Z(i \cdot 2^{-n}, \omega),\ i = 0, 1, \cdots, 2^n\}$, $K_n(\omega)$ be the boundary of $J_n(\omega)$ and F_n the number of vertices of $J_n(\omega)$. The vectors $\{Z(i \cdot 2^{-n}, \omega) - Z((i-1)2^{-n}, \omega)\}\ i = 1, \cdots, 2^n$, satisfy the conditions of Baxter and Barndorff-Nielson, so

$$E\{F_n\} \leqq A_2(\log 2^n)^{d-1} = A_3 n^{d-1}.$$

where A_3 is independent of n.

Let $v(n) = a(n)/h(2 \cdot 2^{-n/6})n^{d-1}$.

By (1) we can choose $a(n)$ so that

$$\lim_{n \to \infty} a(n) = \lim_{n \to \infty} 1/v(n) = 0. \tag{3}$$

Let $\{n_i\}$ be a subsequence for which

$$\Sigma a(n_i) < \infty, \quad \Sigma 1/v(n_i) < \infty. \tag{4}$$

Since $F_n(\omega) > 0$, $\mathrm{Prob}\{F_n(\omega) > v(n)E(F_n(\omega))\} < 1/v(n)$. From (4) then, by use of the Borel-Cantelli lemma, we have, with probability one

$$F_{n_i}(\omega) \leqq v(n_i)E(F_{n_i}(\omega)) \leqq A_3 v(n_i)(n_i)^{d-1} \tag{5}$$

for all but a finite number of i.

For linear Brownian motion, Lévy [4] has shown that, with probability one, and uniformly in t,

$$\limsup_{s \to 0} (Z(t+s) - Z(t))(2s \log 1/s)^{-1/2} = 1.$$

It follows from this that for d-dimensional Brownian motion, we have, uniformly in t, and with probability one

$$\lim_{s \to 0} |Z(t+s) - Z(t)|(2s \log 1/s \log\log 1/s)^{-1/2} = 0.$$

Let $J^* = \{P \mid \mathrm{dist}(P, J_n(\omega) \leqq 2 \cdot 2^{-n} \cdot n \log n\}$. By Lévy's result, $J(\omega) \subset J_n^*(\omega)$

or almost all ω, and for n sufficiently large. So henceforth, we consider only such ω that for large n.

$$J_n(\omega) \subset J(\omega) \subset J_n^*(\omega).$$

Henceforth we suppress the ω.

Let $\mu(A)$ be the surface measure on the unit d dimensional sphere of the normals to K (transferred to the origin) at a set A of K, Ω_d be the total surface measure of the sphere. About each vertex $V(n,i)$ of J_n we construct a solid sphere $S(n,i)$ of radius $2^{-n/6}$. Let B_n be the points of $K(\omega)$ not contained in any of the $S(n,i)$. We wish to show that:

LEMMA : $\mu(B_n) \leqq A_4\, \Omega^{d-1} 2^{-n/3} n^d \log n$.

Proof. Consider the vertex $V(n,i)$. The faces of J_n at $V(n,i)$ form a cone $C(n,i)$ Let the normals to all the supporting planes to $V(n,i)$ of $C(n,i)$ be denoted by $N(n,i)$. Let the normals to K in $K \cap S(n,i)$ be denoted by N^* (n,i). Let those elements of $N(n,i)$ not included in $N^*(n,i)$ be denoted by $M(n,i)$. We wish first to show that the "area" $\mu(\quad)$ of $M(n,i)$ satisfies

$$\mu(M(n,i)) < \Omega_{d-1} 2^{-n/3} n \log n \tag{6}$$

where Ω_{d-1} is the area of the $d-1$ dim. sphere. If $\mu(N(n,i)) < \Omega_{d-1} 2^{-n/3}\, n \log n$ we are already done so we assume the contrary. Let $C^*(n,i) = \{P \mid \mathrm{dist}(P, C(n,i) \leqq 2 \cdot \lambda^{-n/2}\, n \log n\}$ and let $T(n,i) = \{P \in C^*(n,i) \mid \mathrm{dist}\,(P, V(n,i)) = 2^{-n/6}\}$. We form the cone

$$C'(n,i) = \{V(n,i) + tP,\ 0 \leqq t \leqq \infty,\ P \in T(n,i)\}.$$

Let $K(n,i)$ be the points of K not in $C'(n,i)$. Since

$$C'(n,i) \supset J_n, \quad K(n,i) \subset K \cap S(n,i).$$

Hence, if $N^+(n,i)$ are the normals to $K(n,i)$, and $M^*(n,i)$ are those elements of $N(n,i)$ not included in $N^+(n,i)$, to show

$$\mu(M^*(n,i)) < \Omega_{d-1} 2^{-n/3} n \log n \tag{7}$$

will be sufficient to prove (6).

Let $N^0(n,i)$ be the normals to the boundary of $C'(n,i)$. Let P'be a point in the boundary of $C'(n,i)$ in $S(n,i)$, and l' the line through P and $V(n,i)$. From P' we drop a perpendicular to the boundary of $C(n,i)$ meeting it at P, and let the line through P and $V(n,i)$ be l. Let τ be the plane of l and l'. The angle α between l and l' will be approximately

$$\alpha = 2^{-n/2} n \log n / 2^{-n/6} = 2^{-n/3} n \log n.$$

Hence $\mu(N(n,i) - N^0(n,i)) < \Omega_{d-1} \cdot 2^{-n/3} \cdot n \log n$. Let $\tau \cap K = k$, and $K \cap l' = P^*$. Let the normals to k and l' at P^* be n and n'. (See Fig. (1))

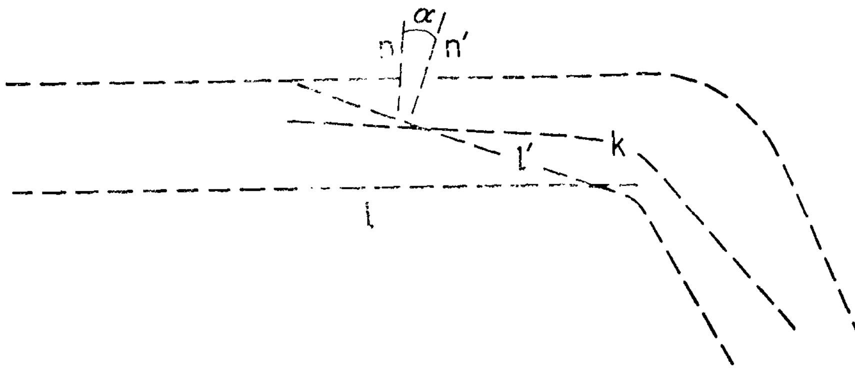

Fig. 1

The normal to l' lies inside that of k, since k must lie inside of $C^*(n,i) \cap \tau$. This implies $N^+(n,i) \supset N^0(n,i)$. Together with the previous inequality we have (7). Summing now over i we have

$$\mu(B_n) \leqq A_4 \Omega^{d-1} F_n 2^{-n/3} n \log n$$

or using the estimate for F_n we have

$$\mu(B_n) \leqq A_4 \Omega_{d-1} 2^{-n/3} n^d \log n.$$

This completes the proof of the lemma.

We define $T_k(\omega) = \cup_{i \geqq k} \cup_j K(\omega) \cap S(n_i, j)$, and let $\mu^*(A) = \mu(A)/\Omega_{d-1}$. This measure will be a probability measure, and from the lemma we have seen that

$$\mu^*\{c(\cup_j K(\omega) \cap S_{i,nj})\} \leqq A_4 2^{-n/3} n_i^d \log n_i$$

where cA indicates the complement of A. Since the right hand side of this inequality is a member of a convergent series, we conclude that $\mu^*(T_k(\omega)) = 1$, by application of the Borel-Cantelli lemma $\mu^*(T(\omega)) = 1$.

Then also for $T(\omega) = \cap_K T_K(\omega)$

$$\mu^*(T(\omega)) =$$

That is, we may take $T(\omega)$ to be a set where the curvature of $K(\omega)$ is concentrated.

Since $K(\omega) \cap S(n_i, j) \subset S(n_i, j)$, we may take the $S(n_i, j)$, $i > k$ to be a covering set. We take $\rho_i = 2 \cdot 2^{-ni/6}$. Then we have, using (5) and the definition of the $v(n)$

$$\begin{aligned} h_{\rho_K}(T(\omega)) &\leqq \sum_{i \geqq K} \Sigma_j h(\operatorname{diam} S(n_i, j)) \\ &= \sum_{i \geqq K} F_{n_i}(\omega) h(2 \cdot 2^{-ni/6}) \\ &\leqq A_4 \sum_{i \geqq K} n_i^{d-1} h(2 \cdot 2^{-ni/6}) v(n_i) \\ &= A_4 \sum_{i \geqq K} a(n_i). \end{aligned}$$

But, since the $\{n, i\}$ were chosen so that the last sequence converges, we have

$$h^*(T(\omega)) = \lim_{i \to \infty} h_{\rho_i}(T(\omega)) = 0.$$

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References

1. O. Barndorff-Nielson and G. Baxter, *Combinational lemmas in n-dimensions*, Trans. Amer. Math. Soc. **108** (1963), 313–325.

2. L. Carleson, *On a class of meromorphic functions and its associated exceptional sets*, Thesis, University of Uppsala, (1950).

3. J. R. Kinney, *The convex hull of plane Brownian motion*, Ann. Math. Statist. **34** (1963), 327–329.

4. P. Lévy, *Processus stochastiques et mouvement Brownien*, Gautier-Villars, Paris, 1948.

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